Consider the following typing derivation trees for the simply-typed λ-calculus extended with pairs, numbers, addition, and Booleans. State whether each of the trees is a valid typing derivation tree. If it is not a valid derivation, briefly explain why. Some typing rules have been numbered for you to make referring to them easier in your answers.

Here are the complete typing rules:

$$\frac{}{\Gamma \vdash \mathit{true} : \texttt{Bool}}\ \text{T-True} \qquad \frac{}{\Gamma \vdash \mathit{false} : \texttt{Bool}}\ \text{T-False} \qquad \frac{}{\Gamma \vdash \mathit{num} : \texttt{Num}}\ \text{T-Num} \qquad \frac{\Gamma(x) = t}{\Gamma \vdash x : t}\ \text{T-Var} \qquad \frac{\Gamma \vdash e_1 : \texttt{Num} \quad \Gamma \vdash e_2 : \texttt{Num}}{\Gamma \vdash e_1 + e_2 : \texttt{Num}}\ \text{T-Plus} \qquad \frac{\Gamma \vdash e_1 : \texttt{Bool} \quad \Gamma \vdash e_2 : t \quad \Gamma \vdash e_3 : t}{\Gamma \vdash \mathit{if}\ e_1\ e_2\ e_3 : t}\ \text{T-If}$$

$$\frac{\Gamma \cup \{x : t_1\} \vdash e : t_2}{\Gamma \vdash \lambda x : t_1 . e : t_1 \to t_2}\ \text{T-Lam} \qquad \frac{\Gamma \vdash e_1 : t_1 \to t_2 \quad \Gamma \vdash e_2 : t_1}{\Gamma \vdash (e_1\ e_2) : t_2}\ \text{T-App} \qquad \frac{\Gamma \vdash e : t_1 \quad \Gamma \vdash e_2 : t_2}{\Gamma \vdash (e_1, e_2) : t_1 * t_2}\ \text{T-Pair} \qquad \frac{\Gamma \vdash e : t_1 * t_2}{\Gamma \vdash \mathit{fst}\ e : t_1}\ \text{T-Fst} \qquad \frac{\Gamma \vdash e : t_1 * t_2}{\Gamma \vdash \mathit{snd}\ e : t_2}\ \text{T-Snd}$$

For brevity in our trees, we will sometimes elide the premises for the T-Var rule. Do not consider this as an invalid usage of this inference rule.

1

$$\frac{\dfrac{\dfrac{}{x : \texttt{Num} \vdash x : \texttt{Num}}\ \text{T-Var}}{\{\} \vdash (\lambda x : \text{Num}.\ x) : \texttt{Num} \to \texttt{Num}}\ \text{T-Abs} \qquad \dfrac{}{\{\} \vdash (\lambda x : \text{Num}.\ x) : \texttt{Num}}\ \text{T-Num}}{\{\} \vdash (\lambda x : \text{Num}.\ x)(\lambda x : \text{Num}.\ x) : \texttt{Num}}\ \text{T-App}$$

2

$$\frac{\dfrac{}{\{\} \vdash \mathit{true} : \texttt{Bool}}\ \text{T-True} \qquad \dfrac{}{\{\} \vdash 1 : \texttt{Num}}\ \text{T-Num} \qquad \dfrac{}{\{\} \vdash \mathit{false} : \texttt{Bool}}\ \text{T-False}}{\{\} \vdash (\mathit{if}\ \mathit{true}\ 1\ \mathit{false}) : \texttt{Num}}\ \text{T-If}$$

3

$$\frac{\dfrac{\dfrac{\dfrac{\dfrac{}{x : \texttt{Num} \vdash x : \texttt{Num}}\ \text{T-Var(1)}}{x : \texttt{Num} \vdash \lambda x : \text{Num}.\ x : \texttt{Num} \to \texttt{Num}}\ \text{T-Lam(2)} \qquad \dfrac{}{x : \texttt{Num} \vdash x : \texttt{Num}}\ \text{T-Var(2)}}{x : \texttt{Num} \vdash ((\lambda x : \text{Num}.\ x)x) : \texttt{Num}}\ \text{T-App(2)}}{\{\} \vdash \lambda x : \text{Num}.\ ((\lambda x : \text{Num}.\ x)x) : \texttt{Num} \to \texttt{Num}}\ \text{T-Lam(1)} \qquad \dfrac{}{\{\} \vdash 5 : \texttt{Num}}\ \text{T-Num}}{\{\} \vdash ((\lambda x : \text{Num}.\ ((\lambda x : \text{Num}.\ x)x))5) : \texttt{Num}}\ \text{T-App(1)}$$

4

$$\frac{\dfrac{\dfrac{}{b : \texttt{Bool}, x : \texttt{Num} \vdash b : \texttt{Bool}}\ \text{T-Var(1)} \qquad \dfrac{\dfrac{\dfrac{}{b : \texttt{Bool}, x : \texttt{Num}, y : \texttt{Num} \vdash x : \texttt{Num}}\ \text{T-Var(2)} \qquad \dfrac{}{b : \texttt{Bool}, x : \texttt{Num}, y : \texttt{Num} \vdash y : \texttt{Num}}\ \text{T-Var(3)}}{b : \texttt{Bool}, x : \texttt{Num}, y : \texttt{Num} \vdash x + y : \texttt{Num}}\ \text{T-Plus(1)}}{b : \texttt{Bool}, x : \texttt{Num} \vdash (\lambda y : \text{Num}.\ x + y) : \texttt{Num} \to \texttt{Num}}\ \text{T-Lam(3)} \qquad \dfrac{\dfrac{\dfrac{}{b : \texttt{Bool}, x : \texttt{Num}, y : \texttt{Num} \vdash y : \texttt{Num}}\ \text{T-Var(4)} \qquad \dfrac{}{b : \texttt{Bool}, x : \texttt{Num}, y : \texttt{Num} \vdash y : \texttt{Num}}\ \text{T-Var(5)}}{b : \texttt{Bool}, x : \texttt{Num}, y : \texttt{Num} \vdash y + y : \texttt{Num}}\ \text{T-Plus(2)}}{b : \texttt{Bool}, x : \texttt{Num} \vdash (\lambda y : \text{Num}.\ y + y) : \texttt{Num} \to \texttt{Num}}\ \text{T-Lam(4)}}{\dfrac{b : \texttt{Bool}, x : \texttt{Num} \vdash (\mathit{if}\ b\ (\lambda y : \text{Num}.\ x + y)\ (\lambda y : \text{Num}.\ y + y)) : \texttt{Num} \to \texttt{Num}}{b : \texttt{Bool} \vdash \lambda x : \text{Num}.\ (\mathit{if}\ b\ (\lambda y : \text{Num}.\ x + y)\ (\lambda y : \text{Num}.\ y + y)) : \texttt{Num} \to (\texttt{Num} \to \texttt{Num})}\ \text{T-Lam(2)}}\ \text{T-If}}{\{\} \vdash \lambda b : \texttt{Bool}.\ \lambda x : \text{Num}.\ (\mathit{if}\ b\ (\lambda y : \text{Num}.\ x + y)\ (\lambda y : \text{Num}.\ y + y)) : \texttt{Bool} \to (\texttt{Num} \to (\texttt{Num} \to \texttt{Num}))}\ \text{T-Lam(1)}$$

5

$$\frac{\dfrac{\dfrac{}{x : \texttt{Num} \vdash x : \texttt{Num}}\ \text{T-Var} \qquad \dfrac{\dfrac{\dfrac{}{x : \texttt{Num} \vdash \mathit{true} : \texttt{Bool}}\ \text{T-True} \qquad \dfrac{}{x : \texttt{Num} \vdash 1 : \texttt{Num}}\ \text{T-Num}}{x : \texttt{Num} \vdash (\mathit{true}, 1) : (\texttt{Bool} * \texttt{Num})}\ \text{T-Prod}}{x : \texttt{Num} \vdash (\mathit{fst}(\mathit{true}, 1)) : \texttt{Num}}\ \text{T-Fst}}{x : \texttt{Num} \vdash x + (\mathit{fst}(\mathit{true}, 1)) : \texttt{Num}}\ \text{T-Plus}}{\{\} \vdash (\lambda x : \text{Num}.\ x + (\mathit{fst}(\mathit{true}, 1))) : \texttt{Num} \to \texttt{Num}}\ \text{T-Lam}$$